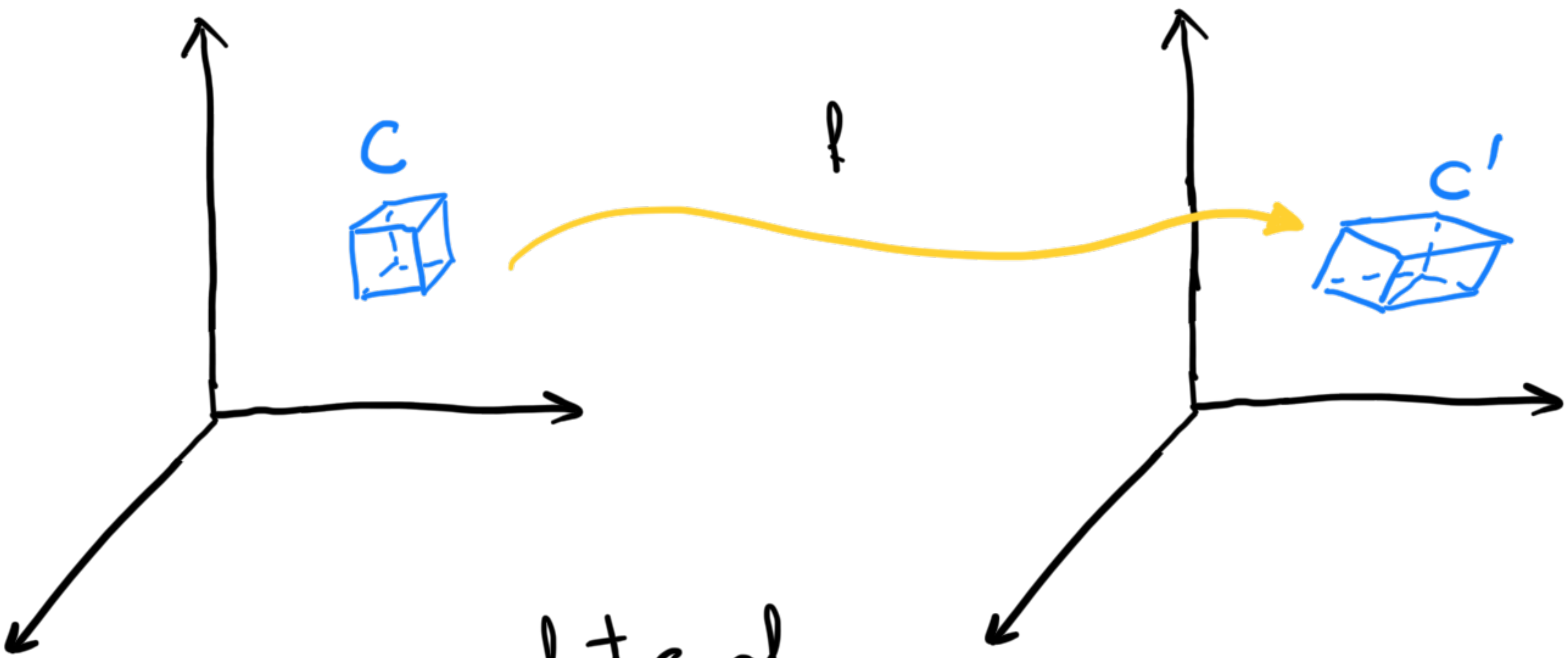
Last time:

det (A) for a square matrix $A \in \mathbb{R}^{n\times n}$

$\parallel$

volume scaling factor of the corresponding $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$



Pick any subset c of and look at $c' = f(c)$

$$\boxed{\det(A) = \pm \frac{\mathrm{Vol}(c')}{\mathrm{Vol}(c)}}$$

does not depend on choice of c

+1 or −1, depending on whether or not f changes orientation (mirror)

Key property: $\det(AB) = (\det A)(\det B)$

gives a practical way to compute det, via Gaussian elimination

- if $\mathrm{REF}(A) = I_n$, then

$$A = \dots\; D_i^{(\lambda)}\; S_{jk}\; T_{LM}^{(\mu)}\; S_{NO}\; D_P^{(\lambda')}\; S_{QR}\; T_{UV}^{(\mu')}\; \dots$$

$$\det A = \dots\; \lambda\; (-1)\; 1\; (-1)\; \lambda'\; (-1)\; 1\; \dots \neq 0$$

- if $\mathrm{REF}(A) = \begin{pmatrix} & * & \\ 0 & 0\;0\;0 & 0 \end{pmatrix}$, then

$$A = \dots\; D_i^{(\lambda)}\; S_{jk}\; T_{LM}^{(\mu)}\; S_{NO}\; \dots \begin{pmatrix} & * & \\ 0 & 0\;0\;0 & 0 \end{pmatrix}$$

$$\det A = \dots\; \lambda\; (-1)\; 1\; (-1)\; \dots\; 0 = 0$$

We conclude that $\det(A) = 0 \iff A$ invertible

Today:

① $A \underset{i \text{ and } j}{\overset{\text{swap row}}{\rightsquigarrow}} A' = S_{ij} A \implies \det(A') = -\det A$

($\det A = 0$ if two rows are proportional)

② $A \underset{\text{row } i \text{ by } \lambda}{\overset{\text{multiply}}{\rightsquigarrow}} A' = D_i^{(\lambda)} A \implies \det(A') = \lambda \det(A)$

$\left(\det(\lambda A) = \lambda^n \det(A) \right)$

③ $A \xrightarrow[\text{to row } j]{\text{add } \lambda \times \text{row } i} A' = T_{ji}^{(\lambda)} A \Longrightarrow \det(A') = \det(A)$

Note: $AB = I_n \overset{\text{converse false}}{\Longrightarrow} \det(A)\det(B) = 1$

$AB = I_n \Leftrightarrow B = A^{-1}$

$\det(A)\det(B) = 1 \Leftrightarrow \det(B) = \dfrac{1}{\det A}$

Examples: $\det\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 0$, because row 2 = 3 × row 1

$$\det\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2 \cdot 1 = 2$$

subtract 3 × row 2 from row 1

Because $\det(A) = \det(A^T)$, all properties above hold with columns instead of rows; for example,

$\det\begin{pmatrix} -1 & 7 & 2 \\ 0 & 6 & 0 \\ 1 & 1 & -2 \end{pmatrix} = 0$ because columns 1 and 3 are proportional

New topic: in general, $\det(A+B) \neq \det(A) + \det(B)$

But there is a scenario in which something like this holds;

Assume matrices A, B, C have the same rows $1, \dots, i-1, i+1, \dots, n$

but the i-th row of C is the sum of the i-th rows of A and B

same   same

$$A = \begin{pmatrix} \boxed{\phantom{a_{11} \cdots a_{1n}}} \\ a_{i1} \cdots\cdots a_{in} \\ \boxed{\phantom{a_{11} \cdots a_{1n}}} \end{pmatrix} \qquad B = \begin{pmatrix} \boxed{\phantom{b_{11} \cdots b_{1n}}} \\ b_{i1} \cdots b_{in} \\ \boxed{\phantom{b_{11} \cdots b_{1n}}} \end{pmatrix} \qquad C = \begin{pmatrix} \boxed{\phantom{a_{i1}+b_{i1} \cdots a_{in}+b_{in}}} \\ a_{i1}+b_{i1} \cdots\cdots a_{in}+b_{in} \\ \boxed{\phantom{a_{i1}+b_{i1} \cdots a_{in}+b_{in}}} \end{pmatrix}$$

same   same

THM 11.1: in setup above, $\det(A) + \det(B) = \det(C)$

(also works for columns instead of rows)

Ex: $$\det\begin{pmatrix} 2 & 5 \\ x & y \end{pmatrix} + \det\begin{pmatrix} 2 & 5 \\ a & b \end{pmatrix} = \det\begin{pmatrix} 2 & 5 \\ x+a & y+b \end{pmatrix}$$

$$2y - 5x \quad + \quad 2b - 5a \quad = \quad 2(y+b) - 5(x+a)$$

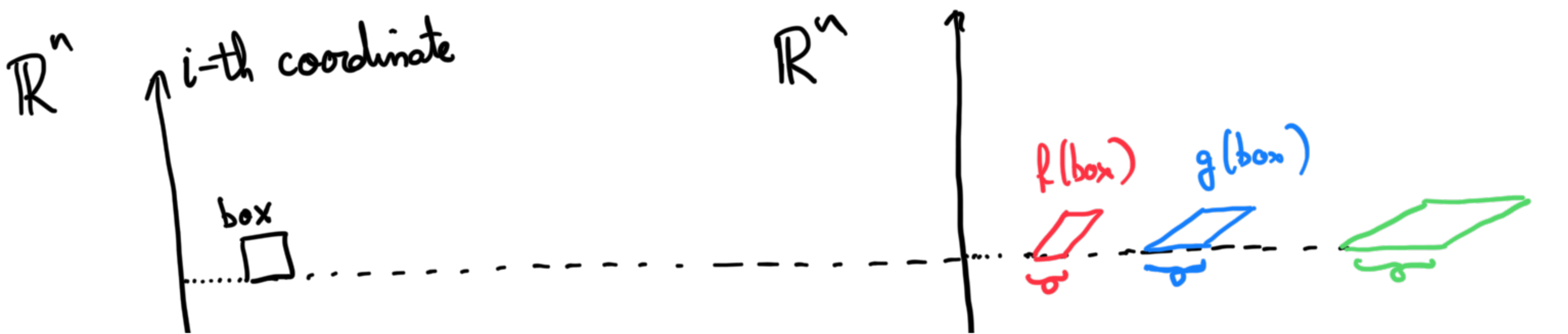
Proof: by geometry (recall that $\det(A) = \dfrac{\mathrm{vol}(f(\mathrm{box}))}{\mathrm{vol}(\mathrm{box})}$)

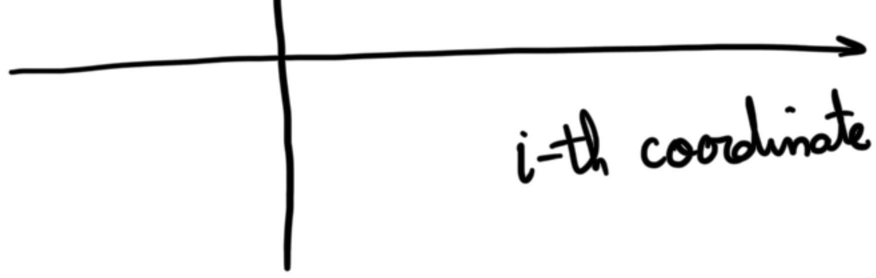
$$A \rightsquigarrow f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(x) = Ax$$

$$B \rightsquigarrow g : \mathbb{R}^n \to \mathbb{R}^n, \quad g(x) = Bx$$

$$C \rightsquigarrow h : \mathbb{R}^n \to \mathbb{R}^n, \quad h(x) = Cx$$

functions f, g, h only differ on i-th coordinate, in which h is the sum of f and g





geometric claim: + = because h is the sum of f and g on i-th coord

$$\Rightarrow \mathrm{vol}(f(\text{box})) + \mathrm{vol}(g(\text{box})) = \mathrm{vol}(h(\text{box}))$$

$$\Rightarrow \frac{\mathrm{vol}(f(\text{box}))}{\mathrm{vol}(\text{box})} + \frac{\mathrm{vol}(g(\text{box}))}{\mathrm{vol}(\text{box})} = \frac{\mathrm{vol}(h(\text{box}))}{\mathrm{vol}(\text{box})}$$

$$\Rightarrow \det(A) + \det(B) = \det(C) \qquad \square$$

Ex: $$\det\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} + \det\begin{pmatrix} 0 & -1 \\ 3 & -1 \end{pmatrix}$$

$$= \underbrace{\det\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}}_{0} + \underbrace{\det\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}}_{0}$$

$$+ \underbrace{\det\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}}_{-2} + \det\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} = -\det\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = -3\cdot(-1)$$

equals to
$0+0+(-2)+3=1$

Generalize above argument to get the Laplace cofactor expansion for $n\times n$ dets

Warm-up: compute determinant of a matrix with a cross of 0's and 1's

i-1 row swaps

$$A = \begin{array}{cc} & \begin{array}{ccccccc} & & & j & & & \end{array} \\ i & \begin{pmatrix} \ddots & & a_{1,j-1} & 0 & a_{1,j+1} & & \ddots \\ & & \vdots & 0 & \vdots & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & 0\ 0\ 0 & 0 & 1 & 0 & 0\ 0\ 0 & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ & & \vdots & 0 & \vdots & & \\ \ddots & & \vdots & 0 & \vdots & & \ddots \\ & & & 0 & & & \\ & & a_{n,j-1} & 0 & a_{n,j+1} & & \end{pmatrix} \end{array} \in \mathbb{R}^{n \times n}$$

j-1 column swaps

Define: $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is just A with the cross $\begin{smallmatrix} & 0 & \\ & 0 & \\ 0\ 0 & 1 & 0\ 0 \\ & 0 & \\ & 0 & \end{smallmatrix}$ removed

Prop: $\det(A) = (-1)^{i+j} \det(A_{ij})$

Proof: after i-1 row swaps and j-1 column swaps, we get

$$\det(A) = (-1)^{i-1+j-1} \det \underbrace{\begin{pmatrix} 1 & 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\ 0 & \\ 0 & \\ 0 & \\ 0 & A_{ij} \\ 0 & \\ 0 & \\ 0 & \end{pmatrix}}$$

enough to show that $\det(B) = \det(A_{ij})$

call this B

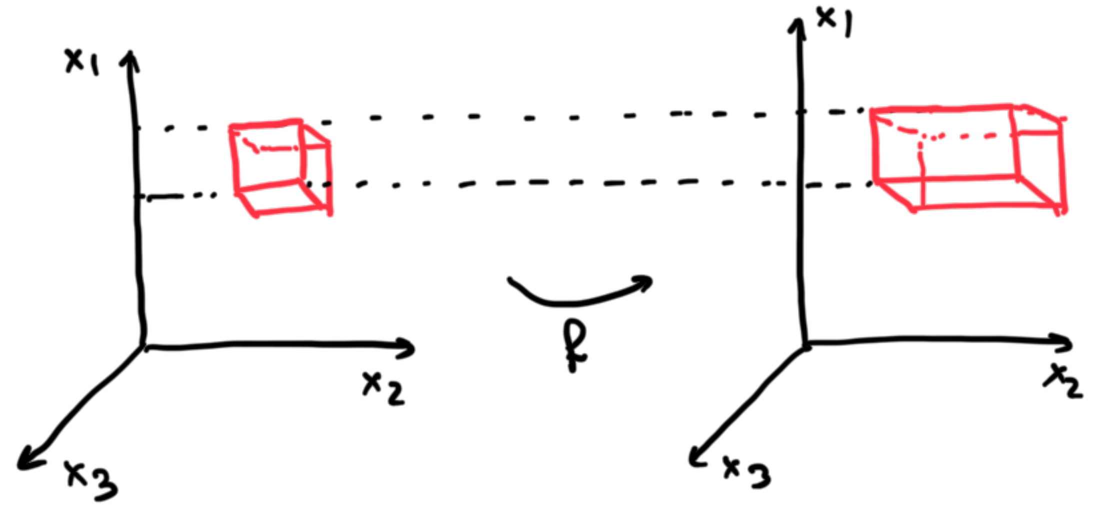
$$f\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = B\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}1&0&0&0&0\\0&&&&\\0&&A_{ij}&&\\0&&&&\\0&&&&\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}x_1\\ A_{ij}\begin{pmatrix}x_2\\ \vdots\\ x_n\end{pmatrix}\end{pmatrix}$$

f leaves the x_1 coordinate untouched, and affects the $x_2, \dots, x_n$ coordinates just like the function

$$g: \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad g\begin{pmatrix}x_2\\ \vdots\\ x_n\end{pmatrix} = A_{ij}\begin{pmatrix}x_2\\ \vdots\\ x_n\end{pmatrix}$$

box in $\mathbb{R}^n$ $\rightsquigarrow$ scaling factor for this box under f $=$ scaling factor for its "top side" under g

(example)



$\Downarrow$

$$\det(B) = \det(A_{ij})$$

$\square$

Ex
$$\det\begin{array}{cc} \begin{array}{ccc} & 2 & \end{array} & \\ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} & 2 \end{array} = (-1)^{2+2} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{array}{cc} \begin{array}{ccc} & & 3 \end{array} & \\ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 1 \\ c & d & 0 \end{pmatrix} & 2 \end{array} = (-1)^{2+3} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = bc - ad$$

THM 11.2 (Laplace cofactor expansion along the i-th row) : $A \in \mathbb{R}$

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \dots + (-1)^{i+n} a_{in} \det(A_{in}).$$

$$A = \begin{pmatrix} & & & \\ a_{i1} & a_{i2} & \cdots\cdots & a_{in} \\ & & & \end{pmatrix}$$

$A_{i1} = A$ without row i, column 1

$A_{i2} = A$ without row i, column 2

$A_{in} = A$ without row i, column n

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^{1+1} a_{11} \det(a_{22}) + (-1)^{1+2} a_{12} \det(a_{21})$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

One can also do cofactor expansion along the i-th column

$$\det(A) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + (-1)^{n+j} a_{nj} \det(A_{nj})$$

$$A = \begin{pmatrix} & j & \\ & a_{1j} & \\ & a_{2j} & \\ & \vdots & \end{pmatrix}$$

A_{1j} is A without row 1, column j

A_{2j} is A without row 2, column j

$\vdots$

A_{nj} is A without row n, column j

Example: 3×3 determinants by cofactor expansion along column 2

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A = (-1)^{1+2} a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + (-1)^{2+2} a_{22} \det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + (-1)^{3+2} a_{32} \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})$$

rearrange terms

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}$$

(Sarrus' rule)

When is cofactor expansion useful? When $\exists$ row or column with many 0's

$$\det\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 0 & 3 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = (-1)^{2+1} 0 \,\_\_ + (-1)^{2+2} 0 \,\_\_ + \underbrace{(-1)^{2+3} 3 \,\_\_}_{=} + (-1)^{2+4} 0 \,\_\_$$

$$(-1)^{2+3} \det\begin{pmatrix} \alpha & \beta & \delta \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Proof of cofactor expansion: as in Prop from first half of class

same   same

same

$$\det\begin{pmatrix} \\ a_{i1} & \cdots & a_{in} \\ \\ \end{pmatrix} = \det\begin{pmatrix} \\ a_{i1} & 0 & 0 & 0 & \cdots & 0 \\ \\ \end{pmatrix} + \cdots + \det\begin{pmatrix} \\ 0 & 0 & \cdots & 0 & a_{in} \\ \\ \end{pmatrix}$$

same   same

row rescale

$$\Rightarrow \det(A) = a_{i1} \cdot \det\begin{pmatrix} \\ 1 & 0 & \cdots & 0 \\ \\ \end{pmatrix} + \cdots + a_{in} \cdot \det\begin{pmatrix} \\ 0 & \cdots & 0 & 1 \\ \\ \end{pmatrix}$$

not same anymore
because columns are missing

$$\underset{\text{operations}}{\overset{\text{row}}{=}} a_{i1} \cdot \det\begin{pmatrix} 0 & \\ 0 & \\ 1 & 0 \cdots 0 \\ 0 & \\ 0 & \end{pmatrix} + \cdots + a_{in} \cdot \det\begin{pmatrix} & 0 \\ & 0 \\ 0 \; 0 \cdots 0 & 1 \\ & 0 \\ & 0 \end{pmatrix}$$

$$\underset{\text{warm-up}}{\overset{\text{Prop from}}{=}} a_{i1} (-1)^{i+1} \det\underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{= A_{i1}} + \cdots + a_{in} (-1)^{i+n} \underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{= A_{in}}$$

$$= (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$$

$\square$